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New braid group representations of the D_2 and D_3 types and their Baxterization

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Abstract. New braid group ($\mathbb{B}_n$) representations of the D_2 and D_3 types are obtained by solving the defining relations of $\mathbb{B}_n$ directly. We discuss a procedure (Baxterization) which allows us to construct their corresponding quantum R matrices.

1. Introduction

The quantum Yang–Baxter equation (QYBE)

$$R_{12}(x)R_{13}(xy)R_{23}(y) = R_{23}(y)R_{13}(xy)R_{12}(x) \tag{1.1}$$

plays a central role in the theory of solvable models in statistical mechanics and quantum field theory [1–5]. If V is a complex vector space and $R(x) \in \mathrm{End}(V\otimes V)$ then $R_{ij}(x) \in \mathrm{End}(V\otimes V\otimes V)$ is a matrix that acts as $R(x)$ on the ith and jth spaces and as the identity on the remaining space; $R(x)$ is referred to as the quantum R matrix and $x \in \mathbb{C}$ is the multiplicative spectral parameter. The QYBE takes various forms. In two-dimensional solvable statistical models the formulation (1.1) is mostly associated with vertex models, while the star-triangle form appears in the interaction-round-a-face models. In $(1+1)$-dimensional field theory, the QYBE takes the form of the factorization equations. Another form of the QYBE which proves useful is

$$(\check{R}(x)\otimes I)(I\otimes\check{R}(xy))(\check{R}(y)\otimes I) = (I\otimes\check{R}(y))(\check{R}(xy)\otimes I)(I\otimes\check{R}(x)) \tag{1.2}$$

with

$$\check{R}(x) = PR(x) \tag{1.3}$$

where $P \in \mathrm{End}(V\otimes V)$ denotes the transposition $u\otimes u' \to u'\otimes u$ and $I \in \mathrm{End}(V)$ is the identity map.

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A large number of solutions have been obtained by solving the factorization equations directly [3, 6–8]. Kulish *et al* initiated a programme [9] in which one obtains quantum R matrices whose classical limit are known solutions (r matrices) of the classical Yang–Baxter equation; quasi-classical solutions of the rational and trigonometric type have been obtained [10–12]. Artin's braid group $\mathbb{B}_n$ provides an interesting connection between solvable models in two-dimensional statistical mechanics and field theory and the theory of knots. The matrix $S \equiv \mathring{R}(0)$ which is in fact a representation of Artin's braid group $\mathbb{B}_2$ and from which representations of $\mathbb{B}_n$ (any n) may be constructed, has been extracted from solvable statistical models at criticality and used to construct link polynomials [3, 13].

In this paper, we approach the problem of finding new solutions of the QYBE by first solving for new braid group representations; this is done by solving the defining relations of $\mathbb{B}_n$ directly. We then proceed to construct their corresponding quantum R matrices; this procedure is known as Baxterization [14].

Our paper is organized as follows. In section 2 we briefly introduce Artin's braid group and some known representations. In sections 3 and 4 new braid group representations of the D_2 and D_3 types are presented. In section 5 we describe the method by which we transform a given matrix S into its corresponding quantum R matrix. In sections 6 and 7 we use this procedure to construct the quantum R matrices corresponding to the solutions of the D_2 and D_3 types given in sections 3 and 4. We conclude with a few remarks.

2. Artin's braid group and some known representations

$\mathbb{B}_n$ [15, 16] is generated by a set of $(n-1)$ generators $g_1, g_2, \ldots, g_{n-1}$ and their inverse subject to the following necessary and sufficient defining relations:

$$g_i g_j = g_j g_i \qquad |i-j| \geq 2 \tag{2.1a}$$

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} . \tag{2.1b}$$

Let V be an n-dimensional vector space and $S \in \mathrm{End}(V \otimes V)$ be an $N^2 \times N^2$ matrix that has an inverse. The following mapping is a representation of $\mathbb{B}_n$:

$$\rho : \mathbb{B}_n \to \mathrm{End}(V^{\otimes n}) \qquad \rho(g_1) = I_1 \otimes \ldots \otimes I_{i-1} \otimes S \otimes I_{i+2} \otimes \ldots \otimes I_n \tag{2.2}$$

where the subscript i means the ith vector space in $V^{\otimes n}$ and S acts in the ith and $(i+1)$th vector spaces. The form of (2.2) ensures the satisfaction of (2.1a); no restriction need be imposed on S. The satisfaction of (2.1b) requires that S be a solution of

$$(S \otimes I)(I \otimes S)(S \otimes I) = (I \otimes S)(S \otimes I)(I \otimes S). \tag{2.3}$$

Our reference point throughout this paper are the solutions of (2.3) which can be extracted from Bazhanov and Jimbo's quasi-classical quantum R matrices [11, 12]. We shall refer to them as the standard solutions. Reshetikhin [17] has shown how these standard solutions can be generated from fundamental irreducible representations of the quantized universal enveloping algebras of simple Lie algebras. The solutions are associated with the direct products

$$\Lambda \otimes \Lambda = \sum_{i=1}^{l} \phi_i \tag{2.4}$$

where Λ is the fundamental irreducible representation of some Lie algebra and l is the number of irreducible representations in the decomposition. Their spectral decomposition and characteristic polynomial $\Delta(\lambda)$ follow the decomposition rule (2.4).

$$S=\sum_{i=1}^{l}\lambda_i\mathbb{P}_i \tag{2.5a}$$

$$\Delta(\lambda)\equiv\det(\lambda I-S)=(\lambda-\lambda_1)^{f_1}\dots(\lambda-\lambda_l)^{f_l} \tag{2.5b}$$

where the λ_i are the distinct eigenvalues, the $\mathbb{P}_i$ the projectors and f_i the dimension of ϕ_i. In addition

$$S(k=1)=P \tag{2.5c}$$

where k is the deformation parameter. An example of such a standard solution is that associated with the fundamental irreducible representation of $\mathrm{sl}(2,\mathbb{C})$

$$\begin{aligned}
&S_1(k)=\text{block diag}(\gamma_1,\gamma_2,\gamma_{-1})\\
&\gamma_1=k \qquad \gamma_2=\begin{pmatrix}0 & 1\\ 1 & k-k^{-1}\end{pmatrix} \qquad \gamma_{-1}=k\\
&\Delta(\lambda;k)=(\lambda-k)^3(\lambda+k^{-1})\\
&m(\lambda;k)=(\lambda-k)(\lambda+k^{-1})
\end{aligned} \tag{2.6}$$

where $m(\lambda;k)$ is the minimal polynomial. S_1 is connected to the six-vertex model and the Jones polynomial [7, 18]. S_1 is the first of an infinite family of solutions corresponding to every irreducible representation of $\mathrm{sl}(2,\mathbb{C})$. The question we raised some time ago and which led to an infinite family of new solutions [19] was the following: is S_1 the only distinct solution with the following block structure?

$$\begin{pmatrix}z_1 & & & 0\\ & 0 & z_2 & \\ & z_2 & z_3 & \\ 0 & & & z_4\end{pmatrix} \qquad z_i\neq 0 \text{ for all } i.$$

The answer is no; there are in fact only two distinct solutions, the second one being

$$\begin{aligned}
&S_2(k)=\text{block diag}(\tilde\gamma_1,\tilde\gamma_2,\tilde\gamma_{-1})\\
&\tilde\gamma_1=k \qquad \tilde\gamma_2=\begin{pmatrix}0 & 1\\ 1 & k-k^{-1}\end{pmatrix} \qquad \tilde\gamma_{-1}=-k^{-1}\\
&\Delta(\lambda;k)=(\lambda-k)^2(\lambda+k^{-1})^2\\
&m(\lambda;k)=(\lambda-k)(\lambda+k^{-1})\\
&S_2(k=1)=P^*\neq P.
\end{aligned} \tag{2.7}$$

S_2 is connected to the free fermion model [7] and to the Alexander–Conway link polynomial [20–23]. Note that although it has the same minimal polynomial, it does not follow the decomposition rule of S_1 and is the k-analogue of a graded permutation operator P^*. We shall refer to such solutions as non-standard solutions. More recently non-standard solutions related to B_2 and C_2 [24] as well as to A_n [23, 25, 26] were reported. In the next two sections, we examine new non-standard solutions of the D_2 and D_3 types.

3. New braid group representations of the D_2 type

3.1. $S_i(k)\times S_j(q)$

Our starting point is the solution of (2.3) extracted from Bazhanov and Jimbo's $D_2^{(1)}$ quantum R matrix which we shall refer to as the standard D_2 solution. Its block structure is

$$S=\text{block diag}(\tau_1,\tau_2,\tau_3,\tau_4,\tau_{-3},\tau_{-2},\tau_{-1})$$

$$\tau_1=z_1 \qquad \tau_2=\begin{pmatrix}0 & z_2\\ z_2 & z_3\end{pmatrix} \qquad \tau_3=\begin{pmatrix}0 & 0 & z_4\\ 0 & z_5 & 0\\ z_4 & 0 & z_6\end{pmatrix}$$

$$\tau_{-1}=z_{17} \qquad \tau_{-2}=\begin{pmatrix}0 & z_{15}\\ z_{15} & z_{16}\end{pmatrix} \qquad \tau_{-3}=\begin{pmatrix}0 & 0 & z_{12}\\ 0 & z_{13} & 0\\ z_{12} & 0 & z_{14}\end{pmatrix} \qquad z_i\neq 0 \text{ all } i \tag{3.1}$$

$$\tau_4=\begin{pmatrix}0 & 0 & 0 & z_7\\ 0 & 0 & z_8 & z_9\\ 0 & z_8 & 0 & z_{10}\\ z_7 & z_9 & z_{10} & z_{11}\end{pmatrix}.$$

Solutions of (2.3) which have the block structure described in (3.1) will be referred to as solutions of the D_2 type. The following direct products are also solutions of this type:

$$S_i(k)\times S_j(q)=\text{block diag}(\tau_1,\tau_2,\tau_3,\tau_4,\tau_{-3},\tau_{-2},\tau_{-1}) \qquad i,j=1,2$$

$$\tau_1=kq \qquad \tau_2=\begin{pmatrix}0 & q\\ q & q(k-k^{-1})\end{pmatrix} \qquad \tau_3=\begin{pmatrix}0 & 0 & \beta\\ 0 & q\alpha & 0\\ \beta & 0 & \beta(k-k^{-1})\end{pmatrix}$$

$$\tau_{-1}=\beta\alpha \qquad \tau_{-2}=\begin{pmatrix}0 & k\\ k & k(q-q^{-1})\end{pmatrix} \qquad \tau_{-3}=\begin{pmatrix}0 & 0 & \alpha\\ 0 & \beta k & 0\\ \alpha & 0 & \alpha(q-q^{-1})\end{pmatrix}$$

$$\tau_4=\begin{pmatrix}0 & 0 & 0 & 1\\ 0 & 0 & 1 & (k-k^{-1})\\ 0 & 1 & 0 & (q-q^{-1})\\ 1 & (k-k^{-1}) & (q-q^{-1}) & (q-q^{-1})(k-k^{-1})\end{pmatrix} \tag{3.2a}$$

$$\alpha=\begin{cases}k & i=1\\ -k^{-1} & i=2\end{cases} \qquad \beta=\begin{cases}q & j=1\\ -q^{-1} & j=2.\end{cases}$$

Their characteristic and minimal polynomials are

$$\begin{aligned}
&\Delta(\lambda;k,q)=(\lambda-kq)^9(\lambda-k^{-1}q^{-1})(\lambda+kq^{-1})^3(\lambda+k^{-1}q)^3 && i,j=1\\
&\Delta(\lambda;k,q)=(\lambda-kq)^6(\lambda-k^{-1}q^{-1})^2(\lambda+kq^{-1})^6(\lambda+k^{-1}q)^2 && i=1,j=2\\
&\Delta(\lambda;k,q)=(\lambda-kq)^6(\lambda-k^{-1}q^{-1})^2(\lambda+kq^{-1})^2(\lambda+k^{-1}q)^6 && i=2,j=1\\
&\Delta(\lambda;k,q)=(\lambda-kq)^4(\lambda-k^{-1}q^{-1})^4(\lambda+kq^{-1})^4(\lambda+k^{-1}q)^4 && i=2,j=2\\
&m(\lambda;k,q)=(\lambda-kq)(\lambda-k^{-1}q^{-1})(\lambda+kq^{-1})(\lambda+k^{-1}q) && i,j=1,2.
\end{aligned} \tag{3.2b}$$

Since S_i has the form $(\rho_1, \rho_2, \rho_{-1})$ where $\rho_{1,2,-1} = \gamma_{1,2,-1}$ or $\tilde{\gamma}_{1,2,-1}$ as given in (2.6) or (2.7), it is easy to establish the identification

$$\tau_1 = \rho_1(k) \times \rho_2(q) \qquad \tau_2 = \rho_2(k) \times \rho_1(q)$$
$$\tau_3 \sim \text{block diag}(\tau_1' \equiv \rho_{-1}(k) \times \rho_1(q), \qquad \tau_2' \equiv \rho_2(k) \times \rho_{-1}(q))$$
$$\tau_{-1} = \rho_{-1}(k) \times \rho_{-1}(q) \qquad \tau_{-2} = \rho_1(k) \times \rho_2(q)$$
$$\tau_{-3} = \text{block diag}(\tau_{-1}' \equiv \rho_1(k) \times \rho_{-1}(q), \tau_{-2}' \equiv \rho_{-1}(k) \times \rho_2(q))$$
$$\tau_4 = \rho_2(k) \times \rho_2(q)$$

there the tilde ˜ means to within rearrangement of rows and columns. Thus the canonical form of D_2-type solutions that are direct products of the S_i $(i = 1, 2)$ is block diag$(\tau_1, \tau_1', \tau_2, \tau_2', \tau_4, \tau_{-2}, \tau_{-2}', \tau_{-1}, \tau_{-1}')$. The standard D_2 solution extracted from Jimbo's formula (3.6) in [12] corresponds to the case $k = q$ and $i, j = 1$; it is equal to $k^2(S_1(k) \otimes S_1(k))$.

The characteristic polynomials in (3.2*b*) indicate that for $k = q$ only $S_1(k) \times S_1(k)$ follow the classical decomposition rule

$$(4) \times (4) = (9) + (6) + (1). \tag{3.3}$$

The remaining three solutions do not follow this decomposition rule and are thus of the non-standard type. We now turn to a two-parameter solution of the D_2 type which is not the result of direct products of S_1 and S_2.

3.2. $\tilde{S}(k, q)$: a new two-parameter solution

This solution was found by solving (2.3) directly. The method used is an extension of the one described in [27] and has already led to solutions associated with non-fundamental irreducible representations [19]; a solution is obtained by solving a minimal set of equations and then verifying, using a symbolic manipulation computer code [28], that the full set of equations (2.3) is satisfied. This new solution which we denote $\tilde{S}(k, q)$ is as follows:

$$\tilde{S}(k, q) = \text{block diag}(\tilde{\tau}_1, \tilde{\tau}_2, \tilde{\tau}_3, \tilde{\tau}_4, \tilde{\tau}_{-3}, \tilde{\tau}_{-2}, \tilde{\tau}_{-1})$$

$$\tilde{\tau}_1 = k \qquad \tilde{\tau}_2 = \begin{pmatrix} 0 & 1 \\ 1 & k - k^{-1} \end{pmatrix} \qquad \tilde{\tau}_3 = \begin{pmatrix} 0 & 0 & kq \\ 0 & -k^{-1} & 0 \\ kq & 0 & k(1-q^2) \end{pmatrix}$$

$$\tilde{\tau}_{-1} = k \qquad \tilde{\tau}_{-2} = \begin{pmatrix} 0 & kq \\ kq & k(1-q^2) \end{pmatrix} \qquad \tilde{\tau}_{-3} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -kq^2 & 0 \\ 1 & 0 & k - k^{-1} \end{pmatrix}$$

$$\tilde{\tau}_4 = \begin{pmatrix} 0 & 0 & 0 & q \\ 0 & 0 & -k & [(1-k^2)(1-q^2)]^{1/2} \\ 0 & -k & 0 & [(1-k^2)(1-q^2)]^{1/2} \\ q & [(1-k^2)(1-q^2)]^{1/2} & [(1-k^2)(1-q^2)]^{1/2} & (2k - k^{-1} - q^2 k) \end{pmatrix} \tag{3.4}$$

$$\tilde{S}(1, 1) \neq P$$
$$\Delta(\lambda; k, q) = (\lambda - k)^8 (\lambda + k^{-1})^4 (\lambda + kq^2)^4$$
$$m(\lambda; k, q) = (\lambda - k)(\lambda + k^{-1})(\lambda + kq^2).$$

It is easily shown that $\tilde{S}(k, q)$ is not a particular case of $S_i \times S_j$ and that in fact it cannot be obtained from any direct product $L_1 \times L_2$ where both L_1 and L_2 have two distinct eigenvalues. The characteristic polynomial indicates that $\tilde{S}(k, q)$ is of the non-standard type (it does not follow the decomposition rule (3.3) of the standard D_2 solution). Note that for $q = i$ the minimal polynomial of $\tilde{S}(k, i)$ is

$$m(\lambda; k, q=i) = (\lambda - k)^2(\lambda + k^{-1})$$

which indicates that $\tilde{S}(k, i)$ is non-diagonalizable. Non-diagonalizable one-parameter solutions can also be obtained from the direct products $S_i(k) \times S_j(q)$. We now turn to new solutions of the D_3 type.

4. New braid group representations of the D_3 type

Our reference point is the standard solution of (2.3) extracted from Bazhanov and Jimbo's $D_3^{(1)}$ quantum R matrix. By solving (2.3) directly we found that in addition to the standard solution there are only three other solutions of the D_3 type. The standard solution is follows:

$$S = \text{block diag}(\pi_1, \pi_2, \pi_3, \pi_4, \pi_5, \pi_6, \pi_{-5}, \pi_{-4}, \pi_{-3}, \pi_{-2}, \pi_{-1})$$

$$\pi_1 = \pi_{-1} = k \qquad \pi_2 = \pi_{-2} = \begin{pmatrix} 0 & 1 \\ & w \end{pmatrix} \qquad \pi_3(\xi) = \pi_{-3}(\xi) = \begin{pmatrix} 0 & 0 & 1 \\ & \xi & 0 \\ & & w \end{pmatrix}$$

$$\pi_4 = \pi_{-4} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ & 0 & 1 & 0 \\ & & w & 0 \\ & & & w \end{pmatrix} \qquad \pi_5(\chi) = \pi_{-5}(\chi) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ & 0 & 0 & 1 & 0 \\ & & \chi & 0 & 0 \\ & & & w & 0 \\ & & & & w \end{pmatrix} \tag{4.1a}$$

$$\pi_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & k^{-1} \\ & 0 & 0 & 0 & k^{-1} & -k^{-1}w \\ & & 0 & k^{-1} & k^{-1}w & k^{-2}w \\ & & & 0 & k^{-1}w & k^{-2}w \\ & & & & k^{-1}w^2 & -k^{-3}w \\ & & & & & (k^{-1}+k^{-3})w^2 \end{pmatrix}$$

with $w \equiv k - k^{-1}$, $\xi = k$ and $\chi = k$; all submatrices $\pi_{\pm i}$ are symmetric. The characteristic and minimal polynomials of S are

$$\begin{aligned} \Delta(\lambda; k) &= (\lambda - k)^{20}(\lambda + k^{-1})^{15}(\lambda - k^{-5}) \\ m(\lambda; k) &= (\lambda - k)(\lambda + k^{-1})(\lambda - k^{-5}). \end{aligned} \tag{4.1b}$$

Note that S decomposes according to the classical decomposition rule $(6) \times (6) =$

(20)+(15)+(1). The other three solutions are non-standard solutions and divide in two equivalence classes; we give a representative of each class. The first non-standard solution is

$$\tilde{S} = \text{block diag}(\tilde{\pi}_1, \tilde{\pi}_2, \tilde{\pi}_3, \tilde{\pi}_4, \tilde{\pi}_5, \tilde{\pi}_6, \tilde{\pi}_{-5}, \tilde{\pi}_{-4}, \tilde{\pi}_{-3}, \tilde{\pi}_{-2}, \tilde{\pi}_{-1})$$

$$\tilde{\pi}_1 = \tilde{\pi}_{-1} = k \qquad \tilde{\pi}_2 = \tilde{\pi}_{-2} = \pi_2 \qquad \tilde{\pi}_3 = \tilde{\pi}_{-3} = \pi_3(\xi = -k^{-1})$$

$$\tilde{\pi}_4 = \tilde{\pi}_{-4} = \pi_4 \qquad \tilde{\pi}_5 = \tilde{\pi}_{-5} = \pi_5(\chi = k)$$

$$\tilde{\pi}_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & k^{-1} \\ & 0 & 0 & 0 & -k & iw \\ & & 0 & k^{-1} & iw & w \\ & & & 0 & iw & w \\ & & & & 2w & -iw \\ & & & & & 0 \end{pmatrix} \tag{4.2a}$$

where $w \equiv k - k^{-1}$ and $\tilde{\pi}_0$ is symmetric. The characteristic and minimal polynomials of $\tilde{S}$ are

$$\begin{aligned} \Delta(\lambda; k) &= (\lambda - k)^{18}(\lambda + k^{-1})^{17}(\lambda - k^{-1}) \\ m(\lambda; k) &= (\lambda - k)(\lambda + k^{-1})(\lambda - k^{-1}). \end{aligned} \tag{4.2b}$$

The second non-standard solutions are as follows:

$$S^* = \text{block diag}(\pi_1^*, \pi_2^*, \pi_3^*, \pi_4^*, \pi_5^*, \pi_6^*, \pi_{-5}^*, \pi_{-4}^*, \pi_{-3}^*, \pi_{-2}^*, \pi_{-1}^*)$$

$$\pi_1^* = \pi_{-1}^* = k \qquad \pi_2^* = \pi_{-2}^* = \pi_2 \qquad \pi_3^* = \pi_{-3}^* = \pi_3(\xi = k)$$

$$\pi_4^* = \pi_{-4}^* = \pi_4 \qquad \pi_5^* = \pi_{-5}^* = \pi_5(\chi = -k^{-1})$$

$$\pi_6^* = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & k^{-1} \\ & 0 & 0 & 0 & k^{-1} & k^{-1}w \\ & & 0 & -k & iw & -ik^{-1}w \\ & & & 0 & iw & -ik^{-1}w \\ & & & & 2w & -k^{-1}w \\ & & & & & k - k^{-3} \end{pmatrix} \tag{4.3a}$$

with $w \equiv k - k^{-1}$ and π_0^* is symmetric. The characteristic and minimal polynomial of S^* are

$$\begin{aligned} \Delta(\lambda; k) &= (\lambda - k)^{19}(\lambda + k^{-1})^{16}(\lambda + k^{-3}) \\ m(\lambda; k) &= (\lambda - k)(\lambda + k^{-1})(\lambda + k^{-3}). \end{aligned} \tag{4.3b}$$

Note that all non-standard solutions do not follow the classical decomposition rule. We now turn to the problem of transforming these solutions into solutions of (1.2).

5. Baxterization

The purpose of this section is mainly to present certain Baxterization formulae which were introduced in [29]. Jimbo has shown [12] that all quasiclassical quantum R

matrices derived in [11, 12] have the following form:

$$\check{R}(x)=\sum_{i=1}^{l}\delta_i(x)\mathbb{P}_i \tag{5.1a}$$

where l is the number of distinct eigenvalues and

$$\delta_i(x)=a_1+b_ix+c_ix^2+\ldots+x^{l-1}. \tag{5.1b}$$

where $a_i, b_i, \ldots$ are constants. In addition, these solutions satisfy the unitarity and initial conditions. Using the quantized universal enveloping algebras associated with these solutions, Jimbo reduces the QYBE to a set of linear equations for $\check{R}(x)$ which he then uses to determine the eigenvalues $\delta_i(x)$

In the light of those results the strategy for constructing a quantum R matrix of the trigonomeric type given a solution S of (2.3) is as follows. Starting with a braid group representation whose spectral decomposition is

$$S=\lambda_1\mathbb{P}_1+\lambda_2\mathbb{P}_2+\ldots+\lambda_l\mathbb{P}_l$$

we seek a quantum R matrix of the form (5.1). The coefficients are determined by imposing the following constraints:

$$\begin{aligned}&\check{R}(x=0)=S\\&\check{R}(x=1)=\nu I \qquad \text{(initial condition)}\\&\check{R}(x)\check{R}(x^{-1})=\eta(x)I \qquad \text{(unitarity condition).}\end{aligned} \tag{5.2}$$

Before proceeding any further, a point should be made clear. We have examined many cases other than the ones discussed in this paper and our experience with this procedure clearly demonstrates that the constraints (5.2) are not sufficient to insure that the matrix $\check{R}(x)$, obtained through such a construction, is a solution of the QYBE (1.3). Based on the many cases examined, we suspect that the formulae given in (5.6) and (5.8) to Baxterize a given braid group representation S with three and four distinct eigenvalues are quite general; however, at this present stage of development we still must verify that the matrix $\check{R}(x)$ obtained is indeed a solution of the QYBE. This test is most easily done using a symbolic manipulation computer code such as SCHOONSHIP [28].

We begin with the case of two distinct eigenvalues. Substituting (5.1) into (5.2) we get

$$\begin{aligned}&a_1=\lambda_1 \qquad a_2=\lambda_2\\&a_1^2+b_1^2=a_2^2+b_2^2\\&a_1b_1=a_2b_2 \qquad a_1+b_1=a_2+b_2.\end{aligned} \tag{5.3}$$

Solving (5.3) we get the following Baxterian formula:

$$\check{R}(x)=(\lambda_1+\lambda_2x)\mathbb{P}_1+(\lambda_2+\lambda_1x)\mathbb{P}_2=S+\lambda_1\lambda_2xS^{-1}. \tag{5.4}$$

By substituting (5.4) into (1.2) it is easily verified that this formula is valid for any S.

Let us now consider the case $l=3$. Substituting (5.1) into (5.2) we get

$$\begin{aligned}&a_1=\lambda_1 \qquad a_2=\lambda_2 \qquad a_3=\lambda_3\\&a_1+b_1+c_1=a_2+b_2+c_2=a_3+b_3+c_3\\&a_1c_1=a_2c_2=a_3c_3 \qquad b_1(a_1+c_1)=b_2(a_2+c_2)=b_3(a_3+c_3)\\&a_1^2+b_1^2+c_1^2=a_2^2+b_2^2+c_2^2=a_3^2+b_3^2+c_3^2.\end{aligned} \tag{5.5}$$

There are many solutions to (5.5); the ones of interest here are those leading to the following formula:

$$\check{R}(x)=\left[\lambda_1+\left(\lambda_2+\frac{\lambda_1\lambda_3}{\lambda_2}\right)x+\lambda_3x^2\right]\mathbb{P}_1+\left[\lambda_2+(\lambda_1+\lambda_3)x+\frac{\lambda_1\lambda_3}{\lambda_2}x^2\right]\mathbb{P}_2$$
$$+\left[\lambda_3+\left(\frac{\lambda_1\lambda_3}{\lambda_2}+\lambda_2\right)x+\lambda_1x^2\right]\mathbb{P}_3$$
$$=\lambda_1\lambda_3x(x-1)S^{-1}+\lambda_3\left(1+\frac{\lambda_1}{\lambda_2}+\frac{\lambda_1}{\lambda_3}+\frac{\lambda_2}{\lambda_3}\right)xI-(x-1)S \tag{5.6}$$

and to the formulae obtained through all possible permutations of the three indices in (5.6); out of the six possibilities only three are distinct.

For the case $l=4$, conditions (3.2) lead to the following set of equations:

$$\begin{aligned}
&a_1=\lambda_1 \qquad a_2=\lambda_2 \qquad a_3=\lambda_3 \qquad a_4=\lambda_4\\
&a_1+b_1+c_1+d_1=a_2+b_2+c_2+d_2=a_3+b_3+c_3+d_3=a_4+b_4+c_4+d_4\\
&a_1^2+b_1^2+c_1^2+d_1^2=a_2^2+b_2^2+c_2^2+d_2^2=a_3^2+b_3^2+c_3^2+d_3^2=a_4^2+b_4^2+c_4^2+d_4^2\\
&a_1b_1+b_1c_1+c_1d_1=a_2b_2+b_2c_2+c_2d_2=a_3b_3+b_3c_3+c_3d_3=a_4b_4+b_4c_4+c_4d_4\\
&a_1c_1+b_1d_1=a_2c_2+b_2d_2=a_3c_3+b_3d_3=a_4c_4+b_4d_4\\
&a_1d_1=a_2d_2=a_3d_3=a_4d_4.
\end{aligned} \tag{5.7}$$

There are several solutions to (5.7); the one of interest leads to the following formula:

$$\check{R}(x)=\sigma_2(x)S^2+\sigma_1(x)S+\sigma_0(x)I+\sigma_{-1}(x)S^{-1} \tag{5.8}$$

with

$$\sigma_2(x)=(\lambda_2\lambda_3)^{-1}(\lambda_4-\lambda_1)^{-1}(\lambda_4\lambda_2-\lambda_1\lambda_3)x(x-1)$$
$$\sigma_1(x)=1-x-(\lambda_2\lambda_3)^{-1}(\lambda_4-\lambda_1)^{-1}[(\lambda_2+\lambda_3)(\lambda_2\lambda_4-\lambda_1\lambda_3)+\lambda_2\lambda_4^2-\lambda_1^2\lambda_3]x(x-1)$$
$$\sigma_0(x)=(\lambda_2\lambda_3)^{-1}(\lambda_4-\lambda_1)^{-1}\{[(\lambda_1\lambda_3+\lambda_1\lambda_4+\lambda_2\lambda_3+\lambda_2\lambda_4)(\lambda_2\lambda_4-\lambda_1\lambda_3)+\lambda_2\lambda_3(\lambda_4^2-\lambda_1^2)$$
$$+\lambda_1\lambda_4(\lambda_3\lambda_4-\lambda_1\lambda_2)]x^2+[\lambda_3^2\lambda_4(\lambda_1+\lambda_2)-\lambda_1\lambda_2^2(\lambda_3+\lambda_4)]x\}$$
$$\sigma_{-1}(x)=\lambda_1\lambda_4x(x-1)\left[x+\left(\frac{\lambda_3-\lambda_2}{\lambda_4-\lambda_1}\right)\right].$$

More detailed discussion of this Baxterization procedure is given [29]. We now turn to the problem of Baxterizing the solutions described in sections 3 and 4.

6. Baxterization of the D_2 and D_3 type solutions

6.1. The six-vertex and free-fermion models and their direct products: Baxterizing using the two distinct eigenvalues formula

We first Baxterize S_1 and S_2 given in equations (2.6) and (2.7). Using formula (5.4) with $\lambda_1=k$ and $\lambda_2=k^{-1}$ we get:

$$\check{R}_1(x;k)=\text{block diag}(\Gamma_1,\Gamma_2,\Gamma_{-1})$$
$$\Gamma_1=k-k^{-1}x \qquad \Gamma_2=\begin{pmatrix} x(k-k^{-1}) & 1-x\\ 1-x & (k-k^{-1})\end{pmatrix} \qquad \Gamma_{-1}=k-k^{-1}x \tag{6.1}$$

$$\check{R}_2(x;k) = \text{block diag}(\tilde{\Gamma}_1, \tilde{\Gamma}_2, \tilde{\Gamma}_{-1})$$
$$\tilde{\Gamma}_1 = k - k^{-1}x \qquad \tilde{\Gamma}_2 = \begin{pmatrix} x(k-k^{-1}) & 1-x \\ 1-x & k-k^{-1} \end{pmatrix} \qquad \tilde{\Gamma}_{-1} = -k^{-1} + kx. \tag{6.2}$$

It is easily shown that the solvable statistical models associated with $\check{R}_1(x;k)$ and $\check{R}_2(x;k)$ are the $6V(I)$ (six-vertex) and $6V(II)$ (free-fermion models described by Sogo *et al* [7].

The following direct products are also solutions of (1.2):

$$\check{R}_i(x;k) \times \check{R}_j(x;q) = \text{block diag}(T_3, T_2, T_1, T_0, T_{-1}, T_{-2}, T_{-3}) \qquad i,j = 1,2$$

$$T_1 = t_1 v_1 \qquad T_2 = \begin{pmatrix} t_2 v_1 & t_3 v_1 \\ t_3 v_1 & t_4 v_1 \end{pmatrix} \qquad T_3 = \begin{pmatrix} t_2 v_5 & 0 & t_3 v_5 \\ 0 & t_5 v_1 & 0 \\ t_3 v_5 & 0 & t_4 v_5 \end{pmatrix}$$

$$T_{-1} = t_5 v_5 \qquad T_{-2} = \begin{pmatrix} t_1 v_2 & t_1 v_3 \\ t_1 v_3 & t_1 v_4 \end{pmatrix} \qquad T_{-3} = \begin{pmatrix} t_5 v_2 & 0 & t_5 v_3 \\ 0 & t_1 v_5 & 0 \\ t_5 v_3 & 0 & t_5 v_4 \end{pmatrix} \tag{6.3}$$

$$T_4 = \begin{pmatrix} t_2 v_2 & t_3 v_2 & t_2 v_3 & t_3 v_3 \\ t_3 v_2 & t_4 v_2 & t_3 v_3 & t_4 v_3 \\ t_2 v_3 & t_3 v_3 & t_2 v_4 & t_3 v_4 \\ t_3 v_3 & t_4 v_3 & t_3 v_4 & t_4 v_4 \end{pmatrix}$$

with

$$t_1 \equiv k - k^{-1}x \qquad t_2 \equiv x(k-k^{-1}) \qquad t_3 \equiv 1-x$$
$$t_4 = k - k^{-1} \qquad t_5 = \alpha - \alpha^{-1}x$$
$$v_1 \equiv q - q^{-1}x \qquad v_2 \equiv x(q-q^{-1}) \qquad v_3 \equiv 1-x$$
$$v_4 = q - q^{-1} \qquad v_5 = \beta - \beta^{-1}x$$

and where α and β are as defined in (3.2*a*). Note that $P[k^2\check{R}_1(x;k) \times \check{R}_1(x;k)]$ is equal to Jimbo's $D_2^{(1)}$ quantum R matrix [12].

6.2. *Quantum R matrices associated with $S_i(k) \times S_j(q)$ and $\tilde{S}(k,q)$: Baxterizing using the three and four distinct eigenvalues formulae*

The direct products $S_i(k) \times S_j(q)$ provide a good testing ground for the Baxterization formulae discussed in section 5. We first consider the case $k = q$ whose three distinct eigenvalues are k^2, -1 and k^{-2}. With the choice $\lambda_1 = k^2$, $\lambda_2 = -1$ and $\lambda_2 = k^{-2}$ formula (5.6) yields $\check{R}_i(x;k) \times \check{R}_j(x;k)$ given in (6.3) illustrating the fact that we can Baxterize S_1 and S_2 first and then take the direct product or take the direct product first and then Baxterize. It is interesting to note that there are two distinct ways of Baxterizing $S_1(k) \times S_1(k)$, each one corresponding to a different ordering of the eigenvalues. While the ordering $\lambda_1 = k^2$, $\lambda_2 = -1$ and $\lambda_3 = k^{-2}$ gives $R_1(x,k) \times R_1(x;k)$ the use of (5.6) with the ordering $\lambda_1 = -1$, $\lambda_2 = k^2$ and $\lambda_3 = k^{-2}$ gives a different quantum R matrix which we denote $\check{R}^*(x;k)$; $-k^4 P\check{R}^*(x;k)$ is equal to the $A_3^{(2)}$ quantum R matrix given by Jimbo in [12]. For all the other direct products, out of the three distinct orderings of the eigenvalues only one leads to a solution of (1.2). Finally we can Baxterize $S_i(k)\times S_j(q)$, which constitutes a four-distinct-eigenvalue case. Choosing $\lambda_1=qk$, $\lambda_2=-qk^{-1}$, $\lambda_3=q^{-1}k^{-1}$ and $\lambda_4=-kq^{-1}$, formula (5.8) gives (6.3) within an overall factor $(1-k^2x)$. We have not examined other orderings of the eigenvalues. The results of the Baxterization of direct products is summarized in figures 1 and 2. We now turn to the Baxterization of the two-parameter solution $\tilde{S}(k, q)$ given in (3.4). Using formula (5.6)

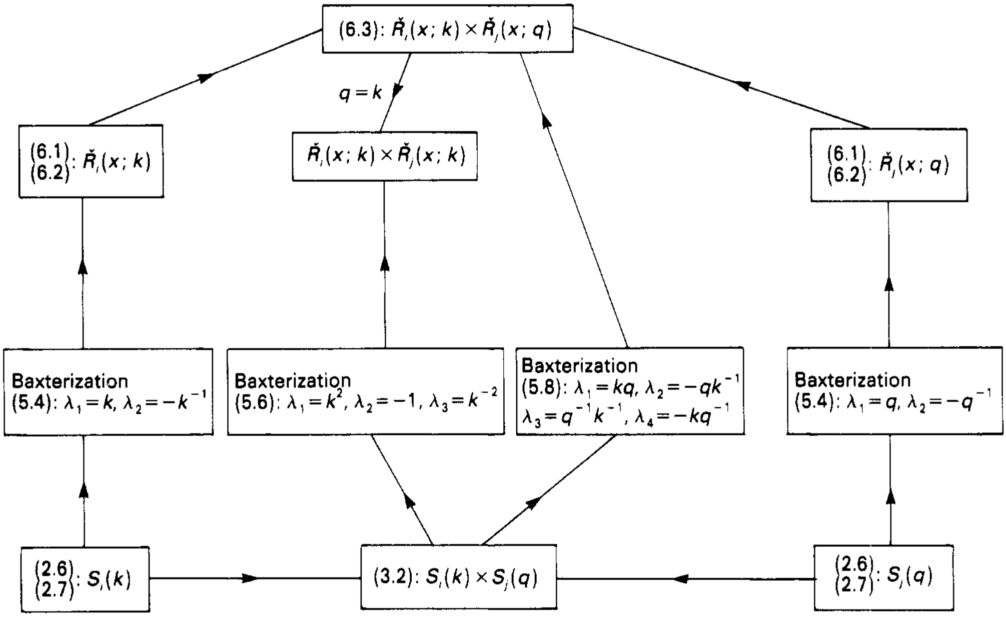

Figure 1. Commutativity of the Baxterization procedure for the direct products $S_i(k)\times S_j(q)$. Numbers in the diagram refers to equations in the text.

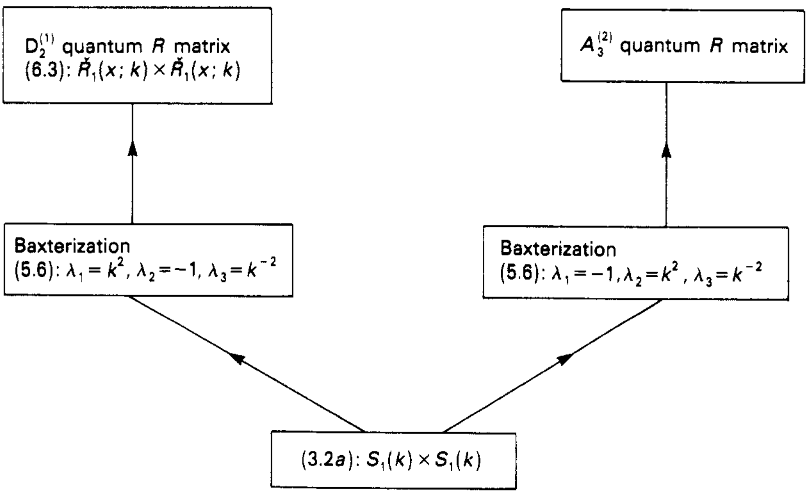

Figure 2. Two distinct ways of Baxterizing $S_1(k)\times S_1(k)$. Numbers in the diagram refer to equations in the text.

with $\lambda_1=-k^{-1}$, $\lambda_2=k$ and $\lambda_3=-kq^2$ we get

$$\check{R}(x;k,q)=\text{block diag}(\tilde{T}_1,\tilde{T}_2,\tilde{T}_3,\tilde{T}_4,\tilde{T}_{-3},\tilde{T}_{-2},\tilde{T}_{-1})$$

$$\tilde{T}_1=(k-k^{-1}x)(1-q^2x) \qquad \tilde{T}_2=\begin{pmatrix} x(k-k^{-1})(1-q^2x) & (1-x)(1-q^2x) \\ (1-x)(1-q^2x) & (k-k^{-1})(1-q^2x) \end{pmatrix}$$

$$\tilde{T}_3=\begin{pmatrix} x(1-q^2)(k-k^{-1}x) & 0 & q(1-x)(k-k^{-1}x) \\ 0 & (1-q^2x)(kx-k^{-1}) & 0 \\ q(1-x)(k-k^{-1}x) & 0 & (1-q^2)(k-k^{-1}x) \end{pmatrix}$$

$$\tilde{T}_{-1}=(k-k^{-1}x)(1-q^2x) \qquad \tilde{T}_{-2}=\begin{pmatrix} x(1-q^2(k-k^{-1}x) & q(1-x)(k-xk^{-1}) \\ q(1-x)(k-xk^{-1}) & (1-q^2)(k-kx^{-1}) \end{pmatrix} \tag{6.4}$$

$$\tilde{T}_{-3}=\begin{pmatrix} (k-k^{-1})x(1-q^2x) & 0 & (1-x)(1-q^2x) \\ 0 & (x-q^2)(k-k^{-1}x) & 0 \\ (1-x)(1-q^2x) & 0 & (k-k^{-1})(1-q^2x) \end{pmatrix}$$

$$T_4=\begin{pmatrix} \omega_1 & \omega_2 & \omega_2 & \omega_3 \\ \omega_2 & \omega_4 & \omega_5 & \omega_6 \\ \omega_2 & \omega_5 & \omega_7 & \omega_6 \\ \omega_3 & \omega_6 & \omega_6 & \omega_8 \end{pmatrix}$$

with

$$\omega_1=x(-k^{-1}x+2q^2xk^{-1}-q^2xk-q^2k^{-1}+k)$$
$$\omega_2=k^{-1}qx(x-1)[(1-q^2)(1-k^2)]^{1/2}$$
$$\omega_3=q(1-x)^2 \qquad \omega_4=(1-q^2)(k-k^{-1})x$$
$$\omega_5=(x-1)(k-k^{-1}q^2x)$$
$$\omega_6=(1-x)[(1-q^2)(1-k^2)]^{1/2} \qquad \omega_7=x(1-q^2)(k-k^{-1})$$
$$\omega_8=(1-x)[k(1-q^2)+k-k^{-1}]+(1-q^2)(k-k^{-1})x.$$

We have verified that (6.4) is a solution of (1.2). Note that the other two distinct orderings, namely $\lambda_1=k$, $\lambda_2=-k^{-1}$, $\lambda_3=-kq^2$ and $\lambda_1=k$, $\lambda_2=-kq^2$, $\lambda_3=-k^{-1}$ do not give solutions of (1.2).

6.3. Baxterization of the D_3 type solutions

The Baxterization of D_3 type solutions is done using (5.6). Using Jimbo's [12] formula (3.6), it may be verified that the Baxterization of the standard solution—given in (4.1) with the orderings $\lambda_1=k$, $\lambda_2=-k^{-1}$, $\lambda_3=k^{-5}$ and $\lambda_1=-k^{-1}$, $\lambda_2=k$, $\lambda_3=k^{-5}$ give the $D_3^{(1)}$ and $A_5^{(2)}$ quantum R matrices respectively; the third possible ordering, namely $\lambda_1=k$, $\lambda_2=k^{-5}$, $\lambda_3=-k^{-1}$ does not lead to a solution of (1.2). We have verified that $\tilde{S}$ given in (4.2) may also be Baxterized into two distinct ways. The quantum R matrix corresponding to the ordering $\lambda_1=k$, $\lambda_2=-k^{-1}$ and $\lambda_3=k^{-1}$ is

$$\check{R}(x;k)=x\tilde{S}^{-1}-\tilde{S} \tag{6.5}$$

while that corresponding to the ordering $\lambda_1=-k^{-1}$, $\lambda_2=k$ and $\lambda_3=k^{-1}$ is

$$\check{R}(x;k)=-k^{-2}x(x-1)\tilde{S}^{-1}+k^{-1}(k^2-k^{-2})xI-(x-1)\tilde{S}. \tag{6.6}$$

7. Concluding remarks

Let us first summarize the main results of this paper. By solving the braid group relations directly, we have found new solutions of the D_2 and D_3 types. These solutions distinguish themselves from the standard solutions by the fact that, although they obey the same weight conservation rule (they have the same zeros), they do not obey the classical decomposition rule of tensor products. The second main result consists in the construction of the associated quantum R matrices and illustrates the fact that in some cases there is more than one way to Baxterize a given braid group representation. We conclude with a few remarks.

Remark 1. The underlying mathematical structure behind the standard solutions of (2.3) is the quantized universal enveloping algebra of simple Lie algebras. The fact that the non-standard solutions do not follow the classical decomposition rule hints at a different type of quantized enveloping algebra. Recently, the mathematical structure behind non-standard solutions of the A_n types has been identified (twisted quantum groups) [21].

Remark 2. Results show that the constraints (5.2) are useful guidelines to construct quantum R matrices but are clearly insufficient to ensure that the resulting matrix will be a solution of the QYBE, as the problem of the orderings of the eigenvalues clearly indicates. Additional constraints might be a way of solving this problem. In that respect the work of Bazhanov might shed some light on this problem. In [11] he shows that a meromorphic function $R(\theta)(x = e^{\rho\theta})$ yields a solution of (1.1) provided it satisfies, in addition to constraints equivalent to (5.2), the properties of authomorphicity (quasi-periodicity) and crossing symmetry. It would be interesting to determine under what conditions our prescription leads to such functions.

Remark 3. We suspect that non-standard solutions of (2.3) exists for B_n, C_n, D_n for all n as well as for the exceptional groups.

Remark 4. Recently the quantum superalgebra $U_q\, \mathrm{osp}(2,2)$ has been described by Deguchi *et al* [30]. The braid group representation they extract from this algebra is a special case of the inverse of our two-parameter solution $\tilde{S}(k, q)$ given in (3.4); indeed, a simple symmetry-breaking transformation of the type described in [27] on $[\tilde{S}(k, q = k^{-1})]^{-1}$ followed by a change of variable $k \to k^{-1}$ will give their result. The implications of our two-parameter solution remain to be explored.

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